Second-order supercavitating hydrofoil theory

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The second-order problem of Helmholtz flow past lifting hydrofoils and symmetric struts has been formulated and solved. The solution involves elementary operations on the known solutions of the first-order problem. The second-order lift and drag coefficients are given in integral form. Results obtained for a flat plate at incidence and a symmetric wedge agree with the exact solutions up to the second order. In terms of quantitative improvements, the present second-order theory predicts a lift coefficient for a flat plate at 45° incidence with an error of 8 %, and a drag coefficient for a symmetric wedge of 50° included angle with an error of 5 %; the corresponding angles at which the linear theory would predict force coefficients incurring the same errors are 5° and 15° respectively.

1. Introduction

In this paper, we are concerned with Helmholtz flow past hydrofoils; the flow may detach both at the leading and the trailing edges, as in the case of lifting supercavitating hydrofoils, or at both sides of the trailing edges of finite thickness, as in the case of symmetric struts. The pressure in the cavity is considered constant and equal to that of the undisturbed stream; the cavity length is infinite. Exact solutions of Helmholtz flow past rectilinear shapes have been given by Rayleigh (see Lamb 1945, p. 102) for a flat plate at incidence, and by Bobyleff (see Lamb 1945, p. 104) for a symmetric wedge. The elegant method of Levi-Civita for curved boundaries (see Milne-Thomson 1950, p. 300), unfortunately, only gives exact solutions for a specified Ω function, whose real part is the direction and whose imaginary part is related to the magnitude of the velocity vector. Iteration or approximation methods must be resorted to for treating either the direct problem of finding the pressure on given shapes, or the inverse problem of finding the shape corresponding to a pressure distribution. In a paper dealing with flows with finite or infinite cavities, Wu (1956) has calculated the hydrodynamic characteristics of a circular arc hydrofoil in Helmholtz motion by truncating the infinite series for the Ω function and satisfying the condition on the slope and radius of curvature at two end-points of the circular arc. With increased speeds of displacement vessels, underwater missiles, and hydrofoil craft, the propeller blades, hydrofoils, and struts involved began operating with long trailing cavities. When the cavitation number, i.e. the non-dimensional pressure difference between the free-stream value and that in the cavity, is small, the hydrofoils and struts may be regarded as in Helmholtz motion. It soon became apparent that the method of Levi-Civita was too complicated to allow quick estimates of performance or to provide the required design information.

Recognizing the fact that the hydrofoils concerned are usually at small angles of incidence and the struts are usually thin, Tulin, in a series of papers (Tulin 1953; Tulin & Burkart 1955; Tulin 1956), has developed the linearized theory for such flows including the case of finite cavitation number for which the cavity is of finite extent. It is well known that for flow past airfoils of zero thickness, the linearized theory yields force coefficients (in this case, only the lift coefficient) which are correct to the second order. This fortunate, though fortuitous, circumstance does not present itself in the case of flow past supercavitating hydrofoils. The force coefficients obtained by applying Tulin's linearized theory are only correct to the first order; these include the cavity drag coefficient which is of the same order of magnitude as the friction drag coefficient. For hydrofoils operating in a seaway, the wave induced angle of incidence may reach such values that linear theory is inadequate; propeller blades operating off the design condition may experience the same increase in the angle of incidence.

In view of these facts, it would then seem necessary to pursue the approximate theory to the second order just to yield the same effectiveness as the linear theory for airfoils. Unlike the second-order theory for airfoils (Lighthill 1951; Van Dyke 1956), in which the main objective is to obtain more accurate pressure distributions, we are interested in obtaining more accurate force coefficients. The cross-coupling between the foil thickness and lifting effects found in airfoils has no counterpart in supercavitating hydrofoils as long as the upper foil surface lies within the cavity.

We follow the procedure of Lighthill (1951) in expanding the velocity components into power series in ϵ, which characterizes the size of disturbances. Through substitution of these series expansions of the velocity components into appropriate boundary conditions, the first- and second-order problems present themselves. Unlike the airfoil case, the boundary-value problems associated with the first- and second-order approximations are quite different; the problem associated with the latter is more difficult. Solutions have been obtained for both the lifting and the symmetric case; these contain integrable singularities at the leading edge except for blunt-nose struts for which edge correction has been applied. Calculations have been carried out for a circular arc hydrofoil at incidence and a symmetric wedge. Results obtained for a flat plate, which is a circular arc of zero curvature, and for a symmetric wedge agree with the exact solutions of Rayleigh and Bobyleff to the second order. In the following sections, only the lifting case is presented in detail; results for the symmetric case are stated.

2. Series expansions of the velocity components

Let a lifting, supercavitating hydrofoil of unit chord with bottom shape $\epsilon f_0(x)$ be in a uniform stream of incompressible, inviscid fluid of velocity U with angle of incidence $\epsilon\alpha$, see figure 1. The small parameter ϵ characterizes the size of the small disturbances. The co-ordinate axes x and y are chosen such that

$$f_0(0) = f_0(1) = 0.$$

Let the upper and the lower cavity shapes be denoted by $\epsilon g_u(x)$ and $\epsilon g_l(x)$ respectively. The velocity components u, v, and the cavity shapes are expanded into power series in ϵ:

$$u = U(1+\epsilon u_1+\epsilon^2 u_2+\epsilon^3 u_3+\ldots), \quad (1)$$

$$v = U(\epsilon v_1+\epsilon^2 v_2+\epsilon^3 v_3+\ldots), \quad (2)$$

$$\epsilon g(x) = \int_0^x \frac{v}{u}\,dx$$
$$= \epsilon g_{u1}(x)+\epsilon^2 g_{u2}(x)+\epsilon^3 g_{u3}(x)+\ldots. \quad (3)$$

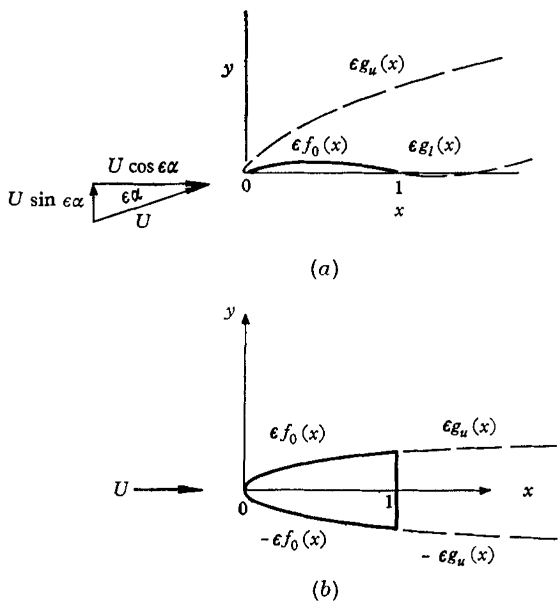

FIGURE 1. Co-ordinate system. (*a*) Lifting hydrofoil; (*b*) symmetric strut.

(Here and subsequently, only the upper cavity shape is treated in detail since the operations on $\epsilon g_l(x)$ are quite similar to those on $\epsilon g_u(x)$.) The velocity components under the integral sign in (3) are evaluated on the cavity boundary. The problem is to find an analytic complex velocity $w(z) = u-iv$, where $z = x+iy$, with singularities admitted at the leading edge and satisfying the following boundary conditions: that the flow be tangent to the foil,

$$\frac{v}{u} = \epsilon f_0'(x), \quad y = \epsilon f_0(x), \quad (0 < x < 1), \quad (4)$$

where the prime denotes differentiation with respect to the argument, that the pressure on the cavity be constant and equal to that of the free stream,

$$ww^* = U^2, \quad \text{on} \quad y = \begin{cases} \epsilon g_u(x), & x > 0, \\ \epsilon g_l(x), & x > 1, \end{cases} \quad (5)$$

where w^* denotes the complex conjugate of w, and that at infinity

$$\left.\begin{aligned} u &= U\cos\epsilon\alpha,\\ v &= U\sin\epsilon\alpha.\end{aligned}\right\} \tag{6}$$

The Kutta condition at the trailing edge requires that $w(1)$ be bounded. In supercavitating flow past struts, the Kutta condition is replaced by the juncture condition, i.e. the slope must be continuous at the body–cavity juncture (Tulin 1953). Other changes to be made in the boundary conditions are quite obvious.

When series expansions for u and v are substituted into the cavity shape expression (3) and the boundary conditions (4) and (5), noting that velocity components away from the x-axis may be obtained by Taylor series expansions about the x-axis, and using a suffix to denote partial differentiation we have

$$\begin{aligned}\epsilon g_u(x) &= \int_0^x \{\epsilon v_1 + \epsilon^2[v_2 - u_1 v_1 + v_{1y} g_{u1}(x)] + O(\epsilon^3)\}\,dx\\ &= \epsilon g_{u1}(x) + \epsilon^2 g_{u2}(x) + O(\epsilon^3), \quad \text{on} \quad y = 0+ \quad (x > 0),\end{aligned} \tag{7}$$

$$\epsilon f_0'(x) = \epsilon v_1 + \epsilon^2[v_{1y} f_0(x) + v_2 - u_1 v_1] + O(\epsilon^3), \quad \text{on} \quad y = 0- \quad (0 < x < 1), \tag{8}$$

and

$$0 = \epsilon 2u_1 + \epsilon^2\left[2u_2 + (v_1^2 + u_1^2) + 2u_{1y}\begin{Bmatrix} g_{u1}(x)\\ g_{l1}(x)\end{Bmatrix}\right] + O(\epsilon^3), \quad \text{on} \quad y = \begin{Bmatrix} 0+, & x > 0,\\ 0-, & x > 1.\end{Bmatrix} \tag{9}$$

The velocity components and their derivatives are evaluated at the x-axis, $y = 0+$ or $0-$ as the case may be. The conditions at infinity become

$$\left.\begin{aligned} 1 + \epsilon u_1 + \epsilon^2 u_2 &= 1 - \tfrac{1}{2}\epsilon^2\alpha^2 + O(\epsilon^3),\\ \epsilon v_1 + \epsilon^2 v_2 &= \epsilon\alpha + O(\epsilon^3).\end{aligned}\right\} \tag{10}$$

To order ϵ, the boundary conditions are

$$\left.\begin{aligned} &v_1 = f_0'(x), \quad y = 0-, \quad 0 < x < 1,\\ &u_1 = 0, \quad y = \begin{Bmatrix} 0+,\\ 0-,\end{Bmatrix} \quad x > \begin{Bmatrix} 0,\\ 1,\end{Bmatrix}\\ &v_1 = \alpha, \quad u_1 = 0 \quad \text{at infinity;}\end{aligned}\right. \tag{11}$$

the upper cavity shape is

$$g_{u1}(x) = \int_0^x v_1\,dx, \quad y = 0+. \tag{12}$$

The linearized problem for symmetric struts has been solved by Tulin (1953), and for lifting hydrofoils, by Tulin & Burkart (1955). The second-order problem, i.e. to order ϵ^2, has the following boundary conditions:

$$v_2 = \frac{d}{dx}[u_1 f_0(x)], \quad y = 0-, \quad (0 < x < 1), \tag{13}$$

$$u_2 = -u_{1y}\begin{Bmatrix} g_{u1}(x)\\ g_{l1}(x)\end{Bmatrix} - \tfrac{1}{2}v_1^2, \quad y = \begin{Bmatrix} 0+,\\ 0-,\end{Bmatrix} \quad x > \begin{Bmatrix} 0,\\ 1,\end{Bmatrix} \tag{14}$$

$$v_2 = 0, u_2 = -\tfrac{1}{2}\alpha^2 \text{ at infinity.} \tag{15}$$

Since by conditions (11), $u_1 = 0$ and by continuity

$$v_{1y} = -u_{1x} = 0 \quad \text{when} \quad x > 0, \quad y = 0+,$$

the second-order upper cavity shape is

$$g_{u2}(x) = \int_0^x v_2\,dx. \tag{16}$$

The first- and the second-order pressure coefficients along the bottom surface of the hydrofoil are easily found to be

$$C_{p1} = -2u_1, \tag{17}$$

and

$$C_{p2} = -2[u_2 + \tfrac{1}{2}(u_1^2 + v_1^2) + u_{1y} f_0(x)]. \tag{18}$$

We note here that the difference between the second-order problem for supercavitating flow past a hydrofoil and that for flow past an airfoil is the absence of conditions on u_2 off the airfoil. This fact makes the second-order problem for airfoils formally the same as the first-order problem, and the formal solution is readily obtainable.

3. Solution of the second-order problem

For the solution of the lifting problem, we first introduce the transformation

$$\zeta = \xi + i\eta = -(z)^{\frac{1}{2}}, \tag{19}$$

where that branch of the complex square root is taken which is positive on $y = 0+$, $x > 0$, and negative on $y = 0-$, $x > 0$. The entire z-plane, except the semi-infinite slit along the positive x-axis, is transformed into the lower half of the ζ-plane. Let now

$$\bar{u}_1(\xi) \equiv u_1[x(\xi)], \text{ etc.}; \tag{20}$$

the boundary conditions (13), and (14) become

$$\bar{v}_2 = \frac{1}{2\xi}\frac{d}{d\xi}[\bar{u}_1 \bar{f}_0(\xi)], \quad \eta = 0-, \quad 0 < \xi < 1 \tag{21}$$

and

$$\bar{u}_2 = -\frac{\bar{v}_{1\xi}}{2\xi}\begin{Bmatrix}\bar{g}_{u1}(\xi)\\ \bar{g}_{l1}(\xi)\end{Bmatrix} - \tfrac{1}{2}\bar{v}_1^2, \quad \eta = 0-, \quad \xi\begin{Bmatrix}< 0,\\ > 1.\end{Bmatrix} \tag{22}$$

The conditions at infinity stay unaltered, and the Kutta condition must be applied at $\zeta = (1, 0-)$. It is noted here that the condition of irrotationality has been used in re-writing equation (14) into (22). The solution is obtained by inspection:

$$\bar{w}_2(\zeta) = \tfrac{1}{2}\bar{w}_1^2 + \frac{1}{\zeta}\frac{d\bar{w}_1}{d\zeta}\int_0^\zeta \zeta\bar{w}_1\,d\zeta + \frac{K}{[\zeta(1-\zeta)]^{\frac{1}{2}}}$$
$$+\frac{1}{\pi}\left(\frac{1-\zeta}{\zeta}\right)^{\frac{1}{2}}\int_0^1 \frac{t^{\frac{1}{2}}\,dt}{(\zeta - t)(1-t)^{\frac{1}{2}}}\left(\frac{\bar{v}_{1t}}{t}\right)\int_0^t s\bar{u}_1(s)\,ds, \tag{23}$$

where

$$K = -\lim_{\zeta\to 1}\frac{(1-\zeta)^{\frac{1}{2}}}{\zeta}\frac{d\bar{w}_1}{d\zeta}\int_0^\zeta \zeta\bar{w}_1\,d\zeta \tag{23a}$$

and Cauchy's principal value of the singular integral is implied. The last term in equation (23) is the complex velocity arising from a slope distribution of

$$-\frac{\bar{v}_{1\xi}}{\xi}\int_0^\xi s\bar{u}_1(s)\,ds$$

along $\eta = 0-$, $0 < \xi < 1$, which satisfies the Kutta condition at $\xi = 1$ and vanishes at infinity. The third term in (23), which is successively imaginary, real, and imaginary along the ξ-axis in the intervals $\xi < 0, 0 < \xi < 1$, and $\xi > 1$ respectively, is the homogeneous solution (Cheng & Rott 1954) which is needed here to satisfy the Kutta condition, thus rendering the solution unique.

It is seen that solution (23) satisfies the boundary condition at infinity,

$$\bar{w}_2(\infty) = -\tfrac{1}{2}\bar{v}_1^2 = -\tfrac{1}{2}\alpha^2,$$

since at infinity $\bar{w}_1(\zeta) \sim -i\alpha + O(|\zeta|^{-\frac{1}{2}})$. The Kutta condition is satisfied by virtue of the third term. In $0 < \xi < 1, \eta = 0-$, the imaginary part of the first two terms of (23)

$$\mathscr{I}\left[\tfrac{1}{2}\bar{w}_1^2+\frac{1}{\zeta}\frac{d\bar{w}_1}{d\zeta}\int_0^\zeta \zeta\bar{w}_1\,d\zeta\right] = -\bar{u}_1\frac{f_0'(\xi)}{2\xi}-\frac{\bar{u}_{1\xi}}{\xi}\int_0^\xi \xi\bar{v}_1\,d\xi-\frac{\bar{v}_{1\xi}}{\xi}\int_0^\xi \xi\bar{u}_1\,d\xi.$$

The last term in the above expression is cancelled by the imaginary part of the last term in (23); condition (21) is satisfied. For $\xi > 1, \eta = 0-$, the last two terms of (23) have no real parts, and the real parts of the first two terms

$$\mathscr{R}\left[\tfrac{1}{2}\bar{w}_1^2+\frac{1}{\zeta}\frac{d\bar{w}_1}{d\zeta}\int_0^\zeta \zeta\bar{w}_1\,d\zeta\right] = -\tfrac{1}{2}\bar{v}_1^2-\frac{\bar{v}_{1\xi}}{\xi}\int_0^\xi \xi\bar{v}_1\,d\xi,$$

which is exactly condition (22). In the same manner, one can readily show that for $\xi < 0, \eta = 0-$, u_2 satisfies (22).

For symmetric struts, the second-order solution is

$$w_{2,s} = \tfrac{1}{2}w_1^2+\frac{dw_1}{dz}\int_0^z w_1\,dz+\frac{K_s}{(1-z)^{\frac{1}{2}}}-\frac{1}{\pi}\left\{\int_0^1\frac{f_0''(t)}{z-t}dt\int_0^t u_1(s)\,ds+\int_1^\infty\frac{(t-1)^{\frac{1}{2}}\,dt}{z-t}\right.$$
$$\left.\times\int_0^1\frac{f_0''(t')\,dt'}{(1-t')^{\frac{1}{2}}(t-t')}\int_0^{t'}u_1(s)\,ds\right\}, \quad (24)$$

where
$$K_s = -\lim_{z\to 1}\left[(1-z)^{\frac{1}{2}}\frac{dw_1}{dz}\int_0^z w_1\,dz\right]. \quad (24a)$$

The last term in (24) is the complex velocity arising from a thickness distribution whose slope is

$$\mp f_0''(x)\int_0^x u_1(s)\,ds, \quad \text{for} \quad 0< x < 1,\ y = 0\pm,$$

whose real part vanishes for $x > 1$, $y = 0\pm$, and satisfies the juncture condition (Tulin 1953). It can be readily shown that solution (24) satisfies all the boundary conditions specified.

The lift coefficient is

$$\begin{aligned}C_L &= \int_0^1 C_p \cos[\epsilon\alpha - \tan^{-1}\epsilon f_0'(x)]\,dx,\\ &= \int_0^1 [\epsilon C_{p1} + \epsilon^2 C_{p2} + O(\epsilon^3)]\{1 - \tfrac{1}{2}\epsilon^2[\alpha - f_0'(x)]^2 + O(\epsilon^4)\}\,dx\\ &= \epsilon\int_0^1 C_{p1}\,dz + \epsilon^2\int_0^1 C_{p2}\,dx + O(\epsilon^3)\\ &= \epsilon C_{L1} + \epsilon^2 C_{L2} + O(\epsilon^3),\end{aligned} \quad (25)$$

and the drag coefficient is

$$\begin{aligned}C_D &= \int_0^1 C_p \sin[\epsilon\alpha - \tan^{-1}\epsilon f_0'(x)]\,dx\\ &= \int_0^1 [\epsilon C_{p1} + \epsilon^2 C_{p2} + O(\epsilon^3)]\{\epsilon[\alpha - f_0'(x)] + O(\epsilon^3)\}\,dx\\ &= \epsilon^2\int_0^1 C_{p1}[\alpha - f_0'(x)]\,dx + \epsilon^3\int_0^1 C_{p2}[\alpha - f_0'(x)]\,dz + O(\epsilon^4)\\ &= \epsilon^2 C_{D1} + \epsilon^3 C_{D2} + O(\epsilon^4).\end{aligned} \quad (26)$$

The first-order force coefficients are known from Tulin & Burkart (1955). The second-order lift coefficient, C_{L2}, is

$$\begin{aligned}C_{L2} &= \int_0^1 C_{p2}\,dx = \int_0^1 2\xi\bar{C}_{p2}\,d\xi\\ &= -4\int_0^1 \xi\left[\bar{u}_1^2 + \frac{\bar{u}_{1\xi}}{\xi}\int_0^\xi t\bar{u}_1(t)\,dt + H(\xi) + G(\xi)\right]d\xi,\end{aligned}$$

where
$$H(\xi) \equiv K\{\xi(1-\xi)\}^{-\frac{1}{2}}, \quad (27)$$

and
$$G(\xi) \equiv \frac{1}{\pi}\left(\frac{1-\xi}{\xi}\right)^{\frac{1}{2}}\int_0^1 \frac{t^{\frac{1}{2}}\,dt}{(\xi - t)(1-t)^{\frac{1}{2}}}\left(\frac{\bar{v}_{1t}}{t}\right)\int_0^t s\bar{u}_1(s)\,ds. \quad (28)$$

Integrating the second term by parts, and noting

$$\lim_{\substack{\xi\to 0\\ \xi\to 1}} \bar{u}_1(\xi)\int_0^\xi 2t\bar{u}_1(t)\,dt = 0,$$

we obtain
$$C_{L2} = -4\int_0^1 \xi[H(\xi) + G(\xi)]\,d\xi. \quad (29)$$

The second-order drag coefficient, C_{D2}, is

$$\begin{aligned}C_{D2} &= \int_0^1 [\alpha - f_0'(x)]\,C_{p2}\,dx\\ &= [\alpha - f_0'(1)]\,C_{L2} + \int_0^1 f_0''(x)\,dx\int_0^x C_{p2}\,dx = [\alpha - f_0'(1)]\,C_{L2}\\ &\quad - 4\int_0^1 \bar{v}_{1\xi}\,d\xi\left\{\bar{u}_1(\xi)\int_0^\xi t\bar{u}_1(t)\,dt + \int_0^\xi t[H(t) + G(t)]\,dt\right\}.\end{aligned} \quad (30)$$

For the symmetric case, the second-order drag coefficient normalized with respect to one-half of the base height is

$$C_{D2,s} = -\frac{2f_0'(1)}{f_0(1)}\int_0^1 [H_s(x)+G_s(x)]\,dx$$
$$-\frac{4}{f_0(1)}\int_0^1 f_0''(x)\left[u_1(x)\int_0^x u_1(t)\,dt+\int_0^x [H_s(t)+G_s(t)]\,dt\right], \tag{31}$$

where
$$H_s(x) \equiv K_s(1-x)^{-\frac{1}{2}}, \tag{32}$$

$$G_s(x) \equiv -\frac{1}{\pi}\left\{\int_0^1 \frac{f_0''(t)\,dt}{x-t}\int_0^t u_1(s)\,ds+\int_1^\infty \frac{(t-1)^{\frac{1}{2}}}{x-t}\,dt\int_0^1 \frac{f_0''(t')\int_0^{t'} u_1(s)\,ds}{(1-t')^{\frac{1}{2}}(t-t')}\,dt'\right\}, \tag{33}$$

in which all integrations are performed along $y = 0+$.

By examining solutions (23) and (24), it is seen that near the leading edge of the hydrofoil, the second-order solution behaves as the square of the first-order solution. The singularities associated with the first-order solution will, then, determine the applicability of the second-order theory, since otherwise a non-integrable pressure may arise. It is known that for supercavitating flow past a lifting hydrofoil, the first-order solution behaves like $x^{-\frac{1}{4}}$; therefore, the second-order pressure is integrable. It may be mentioned here that although the pressure behaves like $x^{-\frac{1}{2}}$ near the leading edge, the suction force is still zero because v_2 behaves like $x^{-\frac{1}{4}}$. The second-order solution for supercavitating flow past a symmetric strut with pointed nose yields an integrable pressure because the first-order solution behaves like $\log x$. For blunt-nose symmetric struts, like the case of blunt-nose airfoils, the $x^{-\frac{1}{2}}$ singularity in the first-order solution renders the second-order solution non-integrable. However, existing results for correcting such singularity behaviour on blunt-nose airfoils (Lighthill 1951) should apply equally well here since the condition at the trailing edge would hardly affect the flow in the neighbourhood of the nose. The uniformly valid pressure coefficient correct to the second order for a strut of nose radius ρ_L is

$$\frac{x}{x+\frac{1}{2}\rho_L}(C_{p1}+C_{p2}).$$

For struts with pointed nose, Van Dyke's (1956) rule to render the approximation near the nose uniformly valid may be applied to find the pressure; it does not, however, contribute to the drag up to the second order. In the lifting case, there exists no such rule. Since the distance from the leading edge to the stagnation point is of order ϵ^4, any edge correction cannot affect the force coefficients which are correct only to ϵ^2.

4. Examples

In this section we present two examples applying the second-order theory; one being flow past a circular arc hydrofoil at incidence, which includes the flat plate as a special case, and the other being flow past a symmetrical wedge. These examples are selected because their exact solutions are known. To facilitate writing, we shall delete the ϵ's in the following.

4.1. *Circular arc*

The equation $f_0(x)$ of a circular arc of small included angle γ, which vanishes at 0 and 1 is

$$f_0(x) = \tfrac{1}{2}\gamma(x)(1-x)+O(\gamma^3). \tag{34}$$

The first-order solution is

$$\bar{w}_1(\zeta) = -[(\alpha-\tfrac{1}{8}\gamma)(1-\zeta)^{\frac{1}{2}}\zeta^{-\frac{1}{2}}+\gamma(\tfrac{1}{2}+\zeta)(1-\zeta)^{\frac{1}{2}}\zeta^{\frac{1}{2}}+i\gamma(\tfrac{1}{2}-\zeta^2)]. \tag{35}$$

The homogeneous solution is easily found to be

$$H(\zeta) = \tfrac{1}{16}\pi\frac{1}{[\zeta(1-\zeta)]^{\frac{1}{2}}}[(\alpha-\tfrac{1}{8}\gamma)^2+\tfrac{33}{16}\gamma(\alpha-\tfrac{1}{8}\gamma)+\tfrac{27}{32}\gamma^2]. \tag{36}$$

Most of the integrals encountered in both the second-order lift and drag expressions are elementary, though sometimes tedious. The integrations of $G(\xi)$ are effected by a change in the order of integration. In the lift expression,

$$\begin{aligned}\int_0^1 \xi G(\xi)\,d\xi &= \frac{1}{\pi}\int_0^1 \xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}}\,d\xi\int_0^1\frac{t^{\frac{1}{2}}\,dt}{(\xi-t)(1-t)^{\frac{1}{2}}}(-\gamma)\int_0^t 2s\bar{u}_1(s)\,ds,\\ &= -\frac{\gamma}{\pi}\int_0^1 t^{\frac{1}{2}}(1-t)^{-\frac{1}{2}}\,dt\int_0^t 2s\bar{u}_1(s)\,ds\int_0^1\frac{\xi^{\frac{1}{2}}(1-\xi)^{\frac{1}{2}}}{\xi-t}\,d\xi,\end{aligned} \tag{37}$$

which is permissible since the residual is zero (Heaslet & Lomax 1954, p. 164). Successive changes in the order of integration, first between ξ and t, then t and t', permits the evaluation of the following integral in the drag expression (30):

$$\begin{aligned}\int_0^1 \xi\,d\xi\int_0^\xi tG(t)\,dt &= -\frac{2\gamma}{\pi}\int_0^1 \xi\,d\xi\int_0^\xi t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt\int_0^1\frac{(t')^{\frac{1}{2}}}{(t-t')(1-t')^{\frac{1}{2}}}\,dt'\int_0^{t'} s\bar{u}_1(s)\,ds,\\ &= -\frac{2\gamma}{\pi}\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}\,dt\int_0^1\frac{(t')^{\frac{1}{2}}\,dt'}{(t-t')(1-t')^{\frac{1}{2}}}\int_0^{t'} s\bar{u}_1(s)\,ds\int_t^1\xi\,d\xi,\\ &= -\frac{\gamma}{\pi}\int_0^1 (t')^{\frac{1}{2}}(1-t')^{-\frac{1}{2}}\,dt'\int_0^{t'} s\bar{u}_1(s)\,ds\int_0^1\frac{(1-t^2)\,t^{\frac{1}{2}}(1-t)^{\frac{1}{2}}}{t-t'}\,dt.\end{aligned} \tag{38}$$

Together with the first-order results, the lift and drag coefficients of a circular arc at incidence are

$$C_L = \tfrac{1}{2}\pi[\alpha(1-\tfrac{1}{4}\pi\alpha)+\tfrac{7}{16}\gamma(1-0{\cdot}323\gamma)-0{\cdot}607\alpha\gamma]+O(\alpha^3), \tag{39}$$

and $$C_D = \tfrac{1}{2}\pi[(\alpha+\tfrac{1}{8}\gamma)^2-\tfrac{1}{4}\pi\alpha^3-0{\cdot}314\alpha^2\gamma-0{\cdot}115\alpha\gamma^2+0{\cdot}016\gamma^3]+O(\alpha^4). \tag{40}$$

It is of interest to compare these with the results of Wu (1956) to the same order of approximation; they are

$$C_L = \tfrac{1}{2}\pi[\alpha(1-\tfrac{1}{4}\pi\alpha)+\tfrac{7}{16}\gamma(1-0{\cdot}386\gamma)-0{\cdot}589\alpha\gamma]+O(\alpha^3),$$

and $$C_D = \tfrac{1}{2}\pi[(\alpha+\tfrac{1}{8}\gamma)^2-\tfrac{1}{4}\pi\alpha^3-0{\cdot}294\alpha^2\gamma-0{\cdot}0675\alpha\gamma^2-0{\cdot}00537\gamma^3]+O(\alpha^4).$$

For a flat plate, $\gamma = 0$, then

$$C_L = \tfrac{1}{2}\pi\alpha(1-\tfrac{1}{4}\pi\alpha),\quad C_D = \tfrac{1}{2}\pi\alpha^2(1-\tfrac{1}{4}\pi\alpha), \tag{41}$$

which agree with those obtained by expanding Rayleigh's result (Lamb 1945, p. 102) to order α^2. The lift and drag coefficients for a circular arc of included angle $\gamma = 16°$, and for a flat plate have been calculated according to equations (39) and (40) and are shown in figures 2 and 3. The results of Wu (1956) for a circular arc hydrofoil and that of Rayleigh for a flat plate together with the linear results are also shown in figures 2 and 3 for the purpose of comparison. It is seen that the second-order correction extends considerably the range of angle of incidence in which the approximate theory predicts very nearly the correct force coefficients.

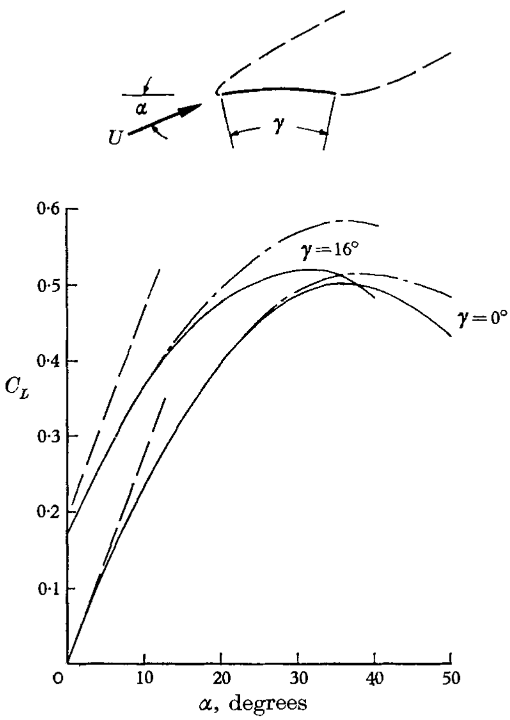

FIGURE 2. Lift coefficient of circular arc hydrofoils. —·—: $\gamma = 0°$, Rayleigh (exact); $\gamma = 16°$, Wu (Levi-Civita's method). ——, Second order; — —, linear.

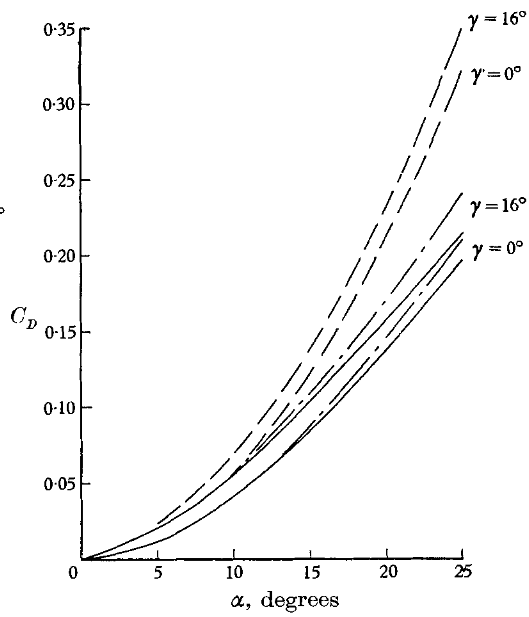

FIGURE 3. Drag coefficient of circular arc hydrofoils. —·—: $\gamma = 0°$, Rayleigh (exact); $\gamma = 16°$, Wu (Levi-Civita's method). ——, Second order; — —, linear.

It is also of interest to examine the pressure along the foils, and the shape of the upper free streamline. To avoid excessive computation, we considered a flat plate at 20° incidence. The pressure distribution, as shown in figure 4, predicted by the second-order theory is very close to the exact result everywhere along the plate except within 4 % chord of the leading edge. As for the cavity shape, figure 5, the difference between the present and the exact results is not discernable to this scale.

4.2. *Symmetric wedge*

We consider a symmetric wedge of included angle 2τ. The first-order solution is†

$$w_1(z) = -\tau\,[2\pi^{-1}\cosh^{-1}(-z^{-\frac{1}{2}}) - i], \tag{42}$$

† This result is easily obtained by first effecting the transformation (19), then the solution to the problem in the ζ-plane is given by Jones & Cohen (1957, p. 17).

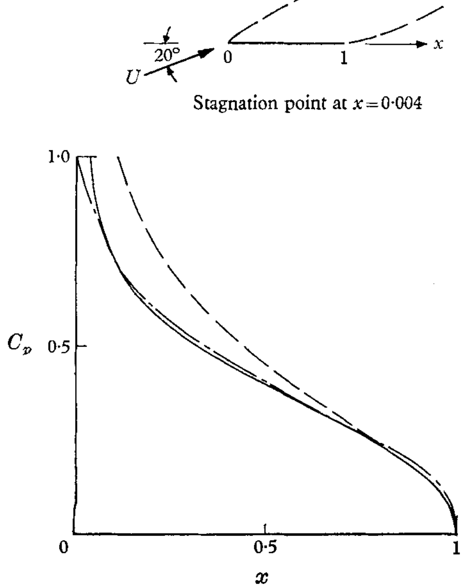

FIGURE 4. Pressure distribution on a flat plate at 20° incidence. —·—, Rayleigh (exact); ——, second order; — —, linear.

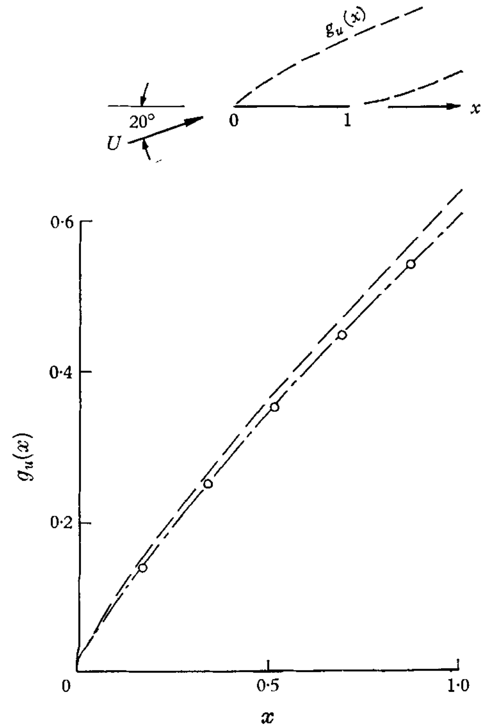

FIGURE 5. Upper cavity shape of a flat plate at 20° incidence. —·—, Rayleigh (exact); ○, second order; — —, linear.

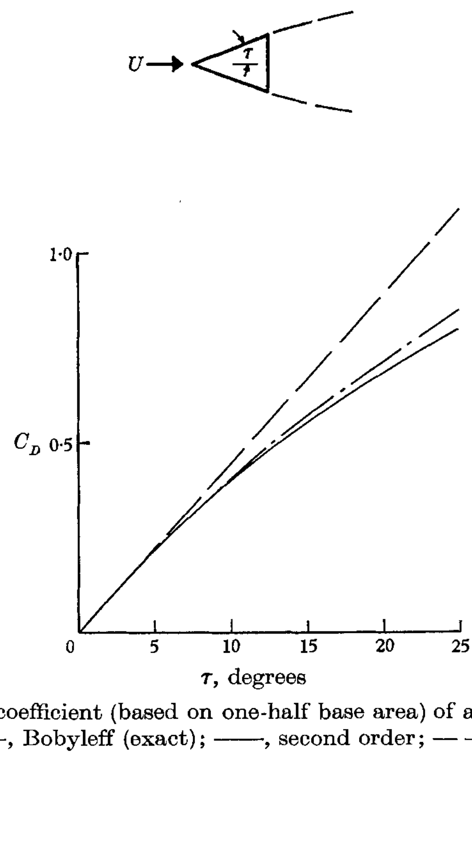

FIGURE 6. Drag coefficient (based on one-half base area) of a symmetrical wedge. —·—, Bobyleff (exact); ——, second order; — —, linear.

in which the sign of $(z)^{\frac{1}{2}}$ is chosen as in (19). The homogeneous solution in the second-order solution is

$$H_s(z) = \frac{2\tau^2}{\pi^2} \frac{1}{(1-z)^{\frac{1}{2}}}. \tag{43}$$

Upon integration, we obtain the drag coefficient normalized with respect to the one-half base height

$$C_D = \frac{8\tau}{\pi}\left[1 - \frac{2}{\pi}\tau\right]. \tag{44}$$

This result agrees to the order τ^2 with the exact solution of Bobyleff (Lamb 1945, p. 104). Figure 6 shows a comparison of the present result with the exact solution and the linear solution.

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REFERENCES

CHENG, H. K. & ROTT, N. 1954 Generalizations of the inversion formula of thin airfoil theory. *J. Rat. Mech. Anal.* **3**, 357.

VAN DYKE, M. D. 1956 Second-order subsonic airfoil theory including edge effects. *NACA TR*, no. 1274.

HEASLET, M. A. & LOMAX, H. 1954 Supersonic and transonic small perturbation theory. *High Speed Aerodynamics and Jet Propulsion*, vol. VI, sect. D. Princeton University Press.

JONES, R. T. & COHEN, D. 1957 Aerodynamics of wings at high speeds, *High Speed Aerodynamics and Jet Propulsion*, vol. VII, sect. A. Princeton University Press.

LAMB, H. 1945 *Hydrodynamics*, 6th ed. Cambridge University Press.

LIGHTHILL, M. J. 1951 A new approach to thin aerofoil theory. *Aero. Quart.* **3**, 193.

MILNE-THOMSON, L. N. 1950 *Theoretical Hydrodynamics*, 2nd ed. London: Macmillan.

TULIN, M. P. 1953 Steady two-dimensional cavity flows about slender bodies. *David Taylor Model Basin Rep.* no. 834.

TULIN, M. P. 1956 Supercavitating flow past foils and struts. *Proceedings of NPL Symposium on Cavitation in Hydrodynamics*. London: H.M. Stationary Office.

TULIN, M. P. & BURKART, M. P. 1955 Linearized theory for flows about lifting foils at zero cavitation number. *David Taylor Model Basin Rep.* no. C-638.

WU, T. Y. 1956 A free streamline theory for two-dimensional fully cavitated hydrofoils. *J. Math. Phys.* **35**, 236.